

# Wilson Loop in Classical Lattice Gauge Theory and the Thermal Width of Heavy Quarkonium

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**Abstract:** We present an estimate for the imaginary part of the recently introduced finite-temperature real-time static potential. It can be extracted from the time evolution of the Wilson loop in classical lattice gauge theory. The real-time static potential determines, through a Schrödinger-type equation and a subsequent Fourier-transform of its solution, the spectral function of heavy quarkonium in finite-temperature QCD. We also compare the results of the classical simulations with those of Hard Thermal Loop improved simulations, as well as with analytic expectations based on resummed perturbation theory.

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## 1. Introduction

Recently, we made an attempt to properly define a static potential in finite-temperature QCD, in the sense of obtaining an object which has a direct connection to the spectral function of the heavy quarkonium system [1]. The spectral function is related to a mesonic correlator obeying a Schrödinger equation in real (Minkowski) time, and the corresponding potential was therefore introduced as the *real-time static potential*. Furthermore, employing resummed perturbation theory, the real-time static potential was shown to develop an imaginary part, which induces a thermal width for the tip of the quarkonium peak observed in the spectral function [2]. In a subsequent work [3], we investigated the extent to which there might be non-perturbative corrections to the imaginary part, utilising classical real-time lattice techniques. The purpose of the current note is to review the results of ref. [3], and also to elaborate on our Hard Thermal Loop improved simulations in some more detail than in ref. [3].

## 2. Real-time static potential

The heavy quarkonium spectral function in the vector channel,  $\rho(\omega)$ , can be obtained using the relation

$$\rho(\omega) = \frac{1}{2} \left( 1 - e^{-\frac{\omega}{T}} \right) \int_{-\infty}^{\infty} dt e^{i\omega t} C_{>}(t, \mathbf{0}) , \quad (2.1)$$

where  $C_{>}(t, \mathbf{0})$  is the mesonic correlator

$$C_{>}(t, \mathbf{r}) \equiv \int d^3\mathbf{x} \left\langle \hat{\psi} \left( t, \mathbf{x} + \frac{\mathbf{r}}{2} \right) \gamma^\mu W \hat{\psi} \left( t, \mathbf{x} - \frac{\mathbf{r}}{2} \right) \hat{\psi}(0, \mathbf{0}) \gamma_\mu \hat{\psi}(0, \mathbf{0}) \right\rangle . \quad (2.2)$$

Here a point-splitting has been introduced to facilitate a perturbative treatment, and  $W$  denotes a Wilson line connecting the adjacent operators along a straight path. The dilepton production rate from  $q\bar{q}$ -annihilation in a quark-gluon plasma is proportional to the thus defined spectral function.

Focusing on infinitely heavy quarks, the correlator can be obtained, up to normalization and a trivial phase factor, from the analytic continuation of a euclidean Wilson loop [1],

$$C_{>}(t, \mathbf{r}) \propto C_E(it, \mathbf{r}) ,$$

$$C_E(\tau, \mathbf{r}) = \frac{1}{N_c} \text{Tr} \langle W(0, \mathbf{r}; \tau, \mathbf{r}) W(\tau, \mathbf{r}; \tau, 0) W(\tau, 0; 0, 0) W(0, 0; 0, \mathbf{r}) \rangle . \quad (2.3)$$

At  $t \neq 0$  we can write the time evolution in the form of a Schrödinger equation,

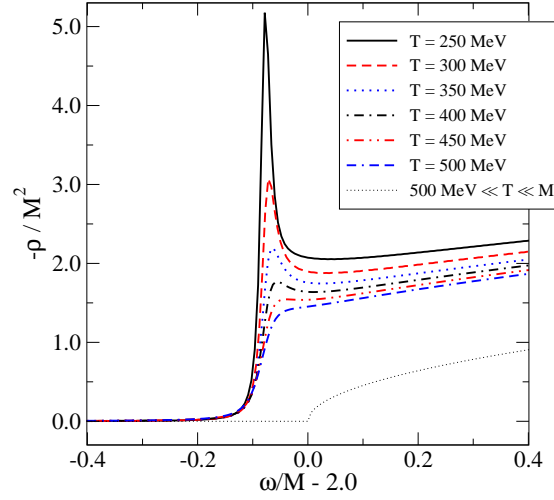
$$[i\partial_t - V_{>}(t, r)] C_{>}(t, \mathbf{r}) = 0 , \quad r \equiv |\mathbf{r}| , \quad (2.4)$$

which defines the object  $V_{>}$  we refer to as the *real-time static potential*.

The simplest estimate for  $V_{>}$  comes from perturbation theory. An analytic computation with proper account taken of HTL-resummation yields the following result in the large-time limit [1]:

$$V_{>}(\infty, r) = -\frac{g^2 C_F}{4\pi} \left[ m_D + \frac{\exp(-m_D r)}{r} \right] - \frac{ig^2 T C_F}{4\pi} \phi(m_D r) ,$$

$$\text{with } \phi(x) = 2 \int_0^\infty \frac{dz z}{(z^2 + 1)^2} \left[ 1 - \frac{\sin(zx)}{zx} \right] . \quad (2.5)$$



**Figure 1:** The resummed perturbative quarkonium contribution ( $M$  denotes the heavy quark pole mass) to the spectral function of the electromagnetic current, in the non-relativistic regime  $(\omega - 2M)/M \ll 1$  [2].

The real part corresponds to the standard Debye-screened potential of a static quark–antiquark pair at finite temperature. The Debye mass is denoted by  $m_D$ . The imaginary part of the potential controls the damping of the correlator  $C_>(t, \mathbf{r})$ , which obeys the Schrödinger equation (2.4).

The spectral function can now be obtained by inserting the static potential into the Schrödinger equation, eq. (2.4), supplemented by the usual mass term and spatial derivatives, and employing subsequently eq. (2.1). The result is shown in fig. 1. The imaginary part of the potential, encoding the Landau damping of the off-shell gluons binding the two heavy quarks together, introduces a thermal width to the tip of the quarkonium peak.

As a next step, we would like to estimate  $V_>(\infty, r)$  beyond perturbation theory. (In principle,  $\rho(\omega)$  could be extracted from lattice Monte Carlo simulations by means of maximum-entropy and related methods; in practice, this involves many subtleties and, possibly, unknown systematic errors. For the current status see, e.g., refs. [4].) A non-perturbative calculation of  $V_>(\infty, r)$  is complicated by the fact that a direct analytic continuation from numerical data for  $C_E(\tau, \mathbf{r})$  is not feasible. It turns out, however, that the imaginary part of  $V_>$  is formally classical [1], and can hence be probed non-perturbatively with classical lattice gauge theory simulations, of the type originally introduced by Grigoriev and Rubakov [5].

### 3. Classical lattice gauge theory simulations

We start our discussion of the real-time lattice techniques by introducing the framework for classical lattice gauge theory simulations [5], which is quite similar to the Kogut-Susskind Hamiltonian approach [6]:

- The fields are discretized using a 3-dimensional spatial lattice. The time coordinate remains continuous.
- Besides the spatial links  $U_i$ , corresponding to the discretized colour-magnetic fields, an electric field  $E_i$  is defined via the relation  $\dot{U}_i(x) = iE_i(x)U_i(x)$ , where  $x \equiv (t, \mathbf{x})$  and  $\dot{U} \equiv \partial U / \partial t$ .

- A temporal gauge is chosen. The space of physical states is constrained to gauge field configurations satisfying the discretized Gauss law,

$$G(x) \equiv \sum_i [E_i(x) - P_{-i}(x)E_i(x - \hat{i})] - j^0(x) \equiv 0, \quad (3.1)$$

with  $j$  denoting a possible colour current, and  $P_i$  the adjoint parallel transporter,  $P_i\phi(x + \hat{i}) = U_i(x)\phi(x + \hat{i})U_i^\dagger(x)$ .

The classical approximation for Yang-Mills fields at finite temperature follows by supplementing the phase space just introduced with a canonical time evolution and an average over initial conditions with a thermal weight. The weight corresponds to the one in the classical partition function,

$$Z = \int \mathcal{D}U_i \mathcal{D}E_i \delta(G) e^{-\beta H}, \quad H = \frac{1}{N_c} \sum_x \left[ \sum_{i < j} \text{Re Tr}(1 - U_{ij}) + \frac{1}{2} \text{Tr}(E_i^2) \right], \quad (3.2)$$

where  $U_{ij}$  is the plaquette. The classical equations of motion for the discretized system can be obtained by invoking the Hamiltonian principle  $\delta S = 0$ , and read  $[\text{Tr}(T^a T^b) = \delta^{ab}/2]$  [7]:

$$\dot{U}_i(x) = iE_i(x)U_i(x), \quad E_i = \sum_a E_i^a T^a, \quad \dot{E}_i^a(x) = -2 \text{Im Tr}[T^a \sum_{|j| \neq i} U_{ij}(x)]. \quad (3.3)$$

A more thorough treatment of the long-range dynamics of hot QCD is possible using the so-called Hard Thermal Loop (HTL) effective theory [8], which is obtained by integrating out the “hard modes” (with momenta of the order of the temperature) from the system, in order to construct an effective theory for the soft modes. To keep the effective theory local, certain on-shell particle degrees of freedom need, however, to be added to the effective Hamiltonian [9]. Once this system is discretized and the classical limit is taken, the properties of the hard modes change, and the associated matching coefficient, denoted by  $m_D^2$ , needs to be tuned correspondingly [10]. In the following we denote the new on-shell particle modes by  $W(x, v)$ . In a numerical implementation the following changes are introduced with respect to the classical setup:

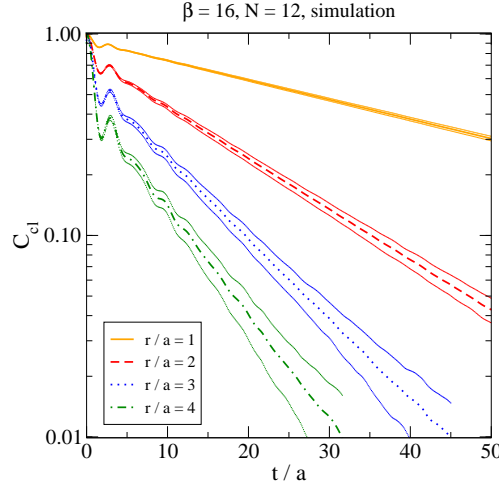
1. The Hamiltonian obtains an additional part,

$$\delta H = \frac{1}{N_c} \sum_x \left[ \int \frac{d\Omega_v}{4\pi} \frac{1}{2} (am_D)^2 \text{Tr}(W^2) \right], \quad (3.4)$$

where  $W \equiv T^a W^a(x, v)$  describes the charge density of the on-shell modes at  $x$  moving in the direction  $v = (1, \mathbf{v})$ .

2. The velocities  $\mathbf{v}$  need to be discretised. This can be done, for instance, with spherical harmonics [11] or with platonic solids [12]. Choosing the latter approach, we can replace  $\int d\Omega_v / 4\pi f(v) \rightarrow 1/N_p \sum_{n=1}^{N_p} f(v_n)$ , where  $N_p$  is the number of vertices of the polyhedron used. The equation of motion of the gauge fields then acquires the source term

$$j(x) = (am_D)^2 \frac{1}{N_p} \sum_{n=1}^{N_p} v_n W_n(x), \quad W_n(x) \equiv W(x, v_n). \quad (3.5)$$



**Figure 2:** Measurement of the correlator  $C_{\text{cl}}(t, r)$  [3]. The corresponding potential is shown in fig. 3.

3. Finally, the new fields also evolve in time, according to the following equation of motion:

$$\dot{W}_n(x) = v_n^i \left( \bar{E}_i(x) - \frac{1}{2} [P_i W_n(x + \hat{i}) - P_{-i} W_n(x - \hat{i})] \right), \quad (3.6)$$

where  $\bar{E}_i(x) \equiv [E_i(x) + P_{-i} E_i(x - \hat{i})]/2$ .

For the purely classical simulations, the required set of initial configurations distributed according to the statistical weight in eq. (3.2) and respecting the Gauss constraint was created using the following algorithm:

1. Pre-generate the spatial gauge links  $U_i$  with a Monte Carlo simulation of the dimensionally reduced effective theory [13].
2. Generate the electric fields from a gaussian distribution [cf. eq. (3.2)].
3. Project onto the space of physical configurations, satisfying the Gauss law [11].
4. Evolve the fields using the EOM, and repeat from step 2, until the fields have thermalized.

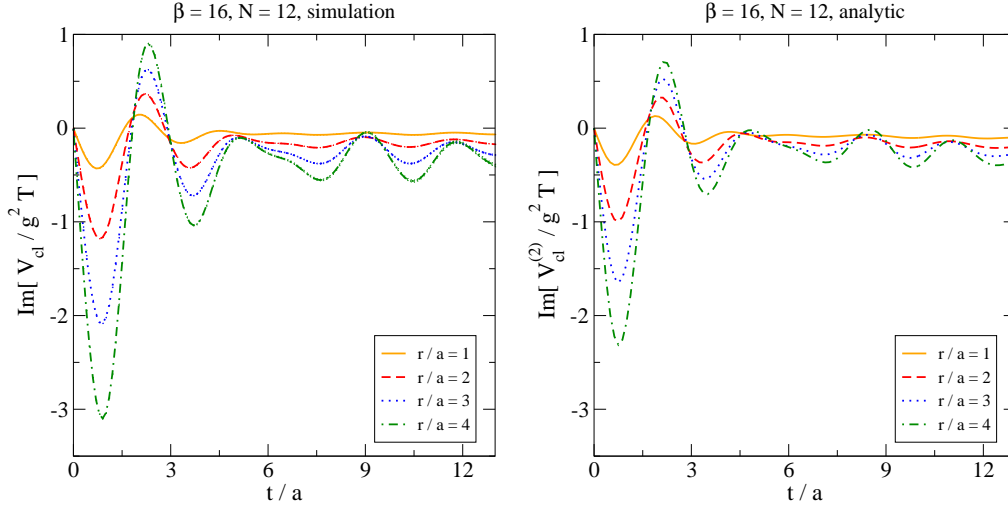
In the HTL-improved case there are minor changes, but the essence of the procedure is the same.

#### 4. The imaginary part of the real-time static potential from Wilson loop dynamics

To obtain the imaginary part of the real-time static potential, a rectangular Wilson loop of spatial extent  $r = |\mathbf{r}|$  and temporal extent  $t$  was measured using classical or HTL-improved simulations. The measured average over a statistical ensemble of initial configurations, as well as over lattice sites and loop orientations, is denoted by  $C_{\text{cl}}(t, r)$  (a typical result is shown in fig. 2). The real-time static potential can then be calculated from eq. (2.4),

$$V_{\text{cl}}(t, r) \equiv \frac{i \partial_t C_{\text{cl}}(t, r)}{C_{\text{cl}}(t, r)}, \quad C_{\text{cl}}(t, r) \equiv \frac{1}{N_c} \text{Tr} \langle W_r^\dagger(t) W_r(0) \rangle, \quad (4.1)$$

with  $W_r(t)$  denoting a spatial Wilson line of length  $r$ . Timelike Wilson lines have disappeared due to the use of temporal gauge. The result for  $V_{\text{cl}}(t, r)$  is purely imaginary, and is shown in fig. 3.



**Figure 3:** The imaginary part of the real-time static potential from the classical simulation (left panel) and from resummed perturbation theory (right panel) [3].

	$\beta$	$N$	$am_D$	confs	$r = 1a$	$r = 2a$	$r = 3a$	$r = 4a$
Simulation	16.0	12	0.0	200	-0.060(2)	-0.156(8)	-0.246(26)	-0.319(56)
	16.0	16	0.0	160	-0.059(2)	-0.155(8)	-0.245(22)	-0.326(48)
	16.0	12	0.211	200	-0.059(2)	-0.147(7)	-0.229(23)	-0.297(51)
	16.0	12	0.350	182	-0.030(2)	-0.064(5)	-0.096(12)	-0.118(21)
	13.5	12	0.250	142	-0.071(2)	-0.174(10)	-0.270(33)	-0.341(97)
Analytic	16.0	$\infty$	0.0	-	-0.0601	-0.1145	-0.1507	-0.1737

**Table 1:** Overview of the results in the large-time limit [3]. The results from the classical and HTL-improved simulations agree within error bars for  $am_D < 0.25$  (at  $\beta = 16$ ).

As seen in fig. 3, the predictions from resummed perturbation theory and from the classical numerical simulations are remarkably similar. At the same time, some amplification of the imaginary part through the inclusion of non-perturbative (and higher-order perturbative) effects is visible in the simulation. The difference between the two results becomes more pronounced at later times. In particular, in the large-time limit, a difference between the perturbative and the numerical results of up to  $\sim 100\%$  can be observed (at  $\beta = 16$ ), cf. table 1.

## 5. Conclusions

The results from the real-time lattice simulations confirm the existence of an imaginary part in the real-time static potential, indicated already by leading-order Hard Thermal Loop resummed perturbation theory. In fact, non-perturbative and higher order perturbative corrections *amplify* the imaginary part, by up to  $\sim 100\%$ . The amplified imaginary part widens (and lowers) the quarkonium peak in fig. 1, although the qualitative structure remains unchanged. As a side remark, we note that the existence of an imaginary part also leads to strong damping in the solution of the Schrödinger equation in eq. (2.4), thus significantly facilitating the numerical determination of the spectral function through eq. (2.1).

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